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AXIALLY SYMMETRIC FLOW

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ОСЕСИММЕТРИЧНЫЙ ПОТОК

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ОСЕСИМЕТРИЧНИЙ ПОТІК

An axially symmetric flow with a free boundary within a sufficiently long domain is investigated. This problem is different from the ones considered earlier, that on part of the free boundary the nonlinear Bernoulli's law is set in kind of inequality. The problem of the existence of boundary value problem is equivalent to minimum problem of some functional with a variable domain of integration. Analyticity and monotony of free boundary are proved. The approximate solution of problem, built by Ritz method, converges to exact solution in L_2 .

Keywords: differential equation, free boundary, functionality, optimization, numerical methods.

Исследуется осесиметричный поток со свободной границей достаточно длинной области. Эта проблема отличается от ранее изученных тем, что на свободной границе нелинейное условие Бернулли задано в виде неравенства. Проблема решения краевой задачи сводится к проблеме минимума функционала с неизвестной областью интегрирования. Доказана аналитичность и монотонность свободной границы. Приближенное решение, построенное методом Ритца, сходится к точному решению в L_2 .

Ключевые слова: дифференциальные уравнения, свободная граница, функционал, оптимизация, численные методы.

Досліджується осесиметричний потік з вільною межею досить довгої області. Ця проблема відрізняється від раніше вивчених тим, що на вільній межі нелінійної умови Бернулли задано у вигляді нерівності. Проблема рішення крайової задачі зводиться до проблеми мінімуму функціонала з невідомою областю інтеграції. Доведена аналітична і монотонність вільної межі. Наближене рішення, побудоване методом Ритца, сходиться до точного рішення в L_2 .

Ключові слова: диференціальні рівняння, вільна межа, функціонал, оптимізація, чисельні методи.

1. Problem. Consider an axially symmetric flow (the x-axis is the axis of symmetry) with vorticity $\omega = const > 0$. The flow is moving from left to right. Then the problem is as follows. Let G be the domain, where

$$\partial G = \Gamma_1 \cup S \cup \Gamma_2 \cup B,\ \Gamma_1 = (x=0,\ 0 \le y \le c),\ \Gamma_2 = (x=a,\ 0 \le y \le b),$$

$$B = (0 \le x \le a, y=0),\ S: y = g(x),\ x \in [0,a];\ g(x) \in C^2[0,a],\ g(0)=c,\ g(a)=b,$$

$c < b,\ g'(0)=0,\ g'(a)=0$ and $g(x)$ is the monotonic function. Let γ be the sufficiently smooth curve, such that point (0, c) is the left end of γ, and γ lies on Γ_2. Curve γ divides the domain G into two subdomains. We designate the lower part of division G by $G_\gamma \subset G$. G_γ is the flow region. We will study the following nonlinear problem. It is necessary to find the stream function $\psi(x,y)$ and the free boundary of γ (G_γ is simply connected domain), where $\psi(x,y)$ satisfies the conditions:

$$\psi_{xx} + \psi_{yy} - y^{-1}\psi_y = \omega y,\ (x,y) \in G_\gamma, \tag{1}$$

$\psi(x,y)$ is the continuous function in $G_\gamma, \psi(x,y)$ is the continuously differentiable function in G_γ, except, perhaps, the point (a, h) ($c < h \le b$, (a, h) is the right end of γ), such that

$$\psi(x,y) = 0,\ (x,y) \in B, \tag{2}$$

$$\psi_x(x,y) = 0,\ (x,y) \in \Gamma_1 \cup \Gamma_2, \tag{3}$$

$$\psi(x,y) = 1,\ (x,y) \in \gamma, \tag{4}$$

$$\psi_x^2(x,y) + \psi_y^2(x,y) \ge v^2 y^2,\ v = const > 0, (x,y) \in \gamma, \tag{5}$$

where on part of γ, lying inside G in (5), there is always an equality. We call (ψ,γ) the classical solution, if (ψ,γ) satisfies (1)-(5).

The problem (1)-(5) differs from problems [1-5], because in condition of (5) there is an inequality when $(x,y) \in \gamma \cap S$.

2. The minimum problem. Let us consider the functional with a variable domain of integration

$$J(\psi,\gamma) = \iint_{G_\gamma} [\psi_x^2 + \psi_y^2 + v^2 y^2 + 2\omega y(\psi - 1)] \frac{dxdy}{y}$$

on set R, where $(\psi,\gamma) \in R : \gamma$ is a Jordan curve such that points (0, c) and (a, b) are the ends of γ. All the points $(x,y) \in \gamma$, except the point (0, c), are above the horizontal $y = c,\ \gamma \in G \cup S$; $\psi(x,y)$ is a continuous function in $\overline{G}_\gamma$, $\psi = 1$ on γ, $\psi = 0$ on B, $\psi(x,y)$ is a continuously differentiable function in G_γ and $J(\psi,\gamma) < \infty$, i.e. J is the bounded functional on R.

Lemma 1. *Let the pair (ψ,γ) be the classical solution of (1)-(5). Then this pair is the stationary point of the functional J or R. Any stationary point of the functional J or R, where γ is a smooth curve, is solution of (1)-(5) (G_γ is the simply connected domain).*

Proof. Let us consider the first variation $J(\psi,\gamma)$ on R [6]. We compute

$$\delta J(\psi, G_\gamma, \delta\psi, \vec{\delta} z) = -2\iint_{G_\gamma}\left[\frac{\partial}{\partial x}\left(\frac{1}{y}\frac{\partial\psi}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial\psi}{\partial y}\right)-\omega\right]\delta\psi\, dxdy + \tag{6}$$

$$+2\int_{\Gamma_1\cup\Gamma_2}\frac{1}{y}\frac{\partial\psi}{\partial n}\delta\psi\, ds + \int_\gamma\left[v^2y^2-\psi_x^2-\psi_y^2\right]\frac{\left(\vec{n},\overrightarrow{\delta z}\right)}{y}ds.$$

Here $\delta\psi$ is the variation of ψ, $\overrightarrow{\delta z} = (\delta x,\ \delta z)$ is variation of *(x, y)*; $\vec{n}$ is the outward normal. Here we will use the act that the first variation satisfies the condition, that $\delta J \geq 0$ for the stationary point (ψ,γ), and $\left(\vec{n},\overrightarrow{\delta z}\right)\leq 0$, if $(x,y)\in\gamma\cap S$. In case if (*a*, *h*) is the right end of γ, where *c*< *h*< *b*, then we will consider a minimum problem for the functional *J* on R_h. If $(\psi,\gamma)\in R_h$, the points (0, *c*) and (*a*, *h*) are the ends of γ. And if $(x,y)\in\gamma$, then $c<y<h$, except the ends of γ: $\psi(x,y)$ have the properties which were considered for $\psi\in R$ (see Theorem 1).

3. Symmetrization. Let us suppose that γ is an admissible curve (i.e. G_γ is the certain domain). First in this section we will show the existence of linear problem (1)-(4) in domain G_γ. This procedure is as follows. Consider set *U*, where $\psi\in U:\psi(x,y)$ is a continuous function, in $\overline{G}_\gamma$, $\psi(x,y)$ is a continuously differentiable function in G_γ, $\psi=0$ on *B*, $\psi=1$ on γ and $L(\psi)<\infty$ (*L* is bounded on *U*),

$$L(\psi)=\iint_{G_\gamma}[\psi_x^2+\psi_y^2+2\omega y(\psi-1)]\frac{dxdy}{y}.$$

Lemma 2. *There is the unique minimum $\psi\in U$ of the functional L in U. This function $\psi(x,y)$ is the classical solution of (1)-(4). If curve γ has the finite length, then*

$$J(\psi,\gamma)=\int_B\left[\frac{1}{y}\psi_y(x.y)\right]\Big|_{y=0}dx+v^2\iint_{G_y}ydxdy+\omega\iint_{G_y}(\psi-1)dxdy.$$

Proof. Here we use variational method and Green's formula (see [7]). Since the solution $\psi(x,y)$ of (1)-(4) satisfies the condition $\psi(x,y)=y^2\alpha(x,y)$, where $\alpha(x,y)$ is a sufficiently smooth function, then

$$\left[y^{-1}\psi_y(x,y)\right]\Big|_{y=0}=2\alpha(x,0).$$

Let us define symmetrization of domain G_γ by Steiner [3] concerning to *x*- and *y*-axes as symmetrization $\Omega=\Pi/G_\gamma$, $\Pi=(0<x<a,\ 0<y<b)$ concerning to *x*-axis and horizontal *y*=*b*. We assume $\psi=1$ for all $(x,y)\in\Omega$.

Lemma 3. *Let $\psi(x,y)$ be the solution of (1)-(4) in domain G_γ, and $\psi^*(x,y)$ is the solution of (1)-(4) in symmetrized domain G^* with the free boundary γ^*. Then $J^*(\psi^*,\gamma^*)\leq J(\psi,\gamma)$, $\psi_x^*(x,y)<0$, $\psi_x^*(x,y)>0$ in G^* and γ^*: x=x(t), y=y(t), $0\leq t\leq T$, where x(t), y(t) are monotonically increasing functions.*

Proof. The idea of the proof is as follows. In case of symmetrization in relation to *x*-axis for every $x_0 \in [0,a]$ we substitute $\psi(x_0,y)$ for monotonically increasing function $\Psi(x_0,y)$, such that

$$mes\{y:\psi(x_0,y)<\rho\} = mes\{y:\Psi(x_0,y)<\rho\},\ \rho\in(0,1).$$

Then we obtain

$$\iint_{G^*}[\Psi_x^2+\Psi_y^2+v^2y^2]\frac{dxdy}{y} \le \iint_{G_\gamma}[\psi_x^2+\psi_y^2+v^2y^2]\frac{dxdy}{y}.$$

The proof of this important inequality has been given in [3], [10]. At last

$$\iint_{G^*}[(\Psi-1)]dxdy = \iint_{G_\gamma}[(\psi-1)]dxdy.$$

This shows that $J^*(\Psi,\gamma^*) \le J(\psi,\gamma)$. Now if we substitute $\Psi(x,y)$ for solution ψ^* of (1)-(4) in G^*, then $J(\psi,\gamma^*) \le J(\Psi,\gamma^*)$ (see Lemma 2) and $\psi^*_y(x,y) > 0$ in G^*. Then it is necessary to carry out symmetrization of G^* in relation to *y*-axis.

4. Theorem of existence. There is a minimal sequence $(\psi_n,\gamma_n)\in R,\ n\in N$. The domain G_n has the free boundary γ_n: $x=x_n(t),\ y=y_n(t),\ 0\le t\le T$, where $x_n(t)$, $y_n(t)$ are monotonically increasing functions (see Lemma 3). On the other hand, according to Lemma 2, $\psi_n(x,y)$ is the solution of (1)-(4) in G_n for every $n\in \mathrm{N}$. If the coordinate system (*x*, *y*) is turned by angle $3\pi/4$ in positive direction, we obtain the system of coordinates (ξ,η) where γ_n: $\eta_n=\eta_n(\xi)$, and $|\eta_n(\xi_1)-\eta_n(\xi_2)|\le|\xi_1-\xi_2|$ for every $n\in N$. Thus, there exists the limit monotonic curve γ. However, γ can also have a common segment with Γ_2 or with horizontal *y=c*.

Let us suppose that

$$1-\frac{\omega b^3}{3}>0, \tag{7}$$

then according to the maximum principle $0\le y^2(1-\omega b^3/3)/b^2+\omega y^3/3\le\ \le\psi_n(x,y)\le 1$ in $\bar{G}_n$ and $\psi_n(x,y)\le y^2(1-\omega c/3)/c^2+\omega y^3/3,\ (x,y)\in\bar{\Pi}_0$, where $\Pi_0=(0<x<a,\ 0<y<c)$. The function $\psi_n(x,y)$ is subharmonic in G_n and $\psi_n(x,y)=\xi_n(x,y)+\omega y^3/3$ (where $\varphi_{nx}=y^{-1}\xi_{ny},\ \varphi_{ny}=y^{-1}\xi_{nx}$), $\varphi_n(x,y)$ is the harmonic function in G_n. It follows that there is function $\psi(x,y)$ such that $\nabla\psi_n\to\nabla\psi$ in $C(\bar{G}_0),\psi_n\to\psi$ in $C(\bar{G}_0)$ (here $\bar{G}_0=G_0+\partial G_0$, i.e. $\bar{G}_0$ is a closed set) for every $G_0\subset G_\gamma$, such that $\bar{G}_0\cap\gamma=\theta$. Then we obtain that ψ satisfies (1)-(3) in limit domain G_γ, and $J(\psi,\gamma)<\infty$.

Let $z_0=x_0+iy_0$ be an interior point of γ. Now let us choose the ray *l* with the top in z_0 such that the angle between *l* and the *x*-axis equals $3\pi/4$. Let us make the cut along *l*.

Then we use inequality

$$|\psi_n(z)-1| \le A\,\mathrm{Re}\left[e^{i\pi/4}(z-z_n)\right]^{1/2},\ z_n \in \gamma_n,\ \lim_{n\to\infty} z_n = z_0$$

for every $z \in \overline{G}_n$ and sufficiently large number A>0. Let us assume that z is an interior point G_γ. Letting $n \to \infty$, and then $z \to z_0$, we conclude that $\psi = 1$ on γ. Thus (ψ, γ) satisfies (1)-(4) in G_γ and

$$J(\psi,\gamma) = 2\int_b \alpha(x,0)dx + v^2 \iint_{G_y} ydxdy + \omega \iint_{G_y} (\psi - 1)dxdy,$$

where $\psi(x,y) = y^2\alpha(x,y)$. It was shown in Lemma 2.

Since $\left(\psi_{ny}y^{-1}\right)\big|_{y=0} = 2\alpha_n(x,0) = \varphi_{nx}(x,0) \Rightarrow \left(\psi_y y^{-1}\right)\big|_{y=0} = 2\alpha(x,0) = \varphi_x(x,0)$ in $C[0,a]$, where $\psi_n(x,y) = y^2\alpha_n(x,y)$ *and* $\varphi(x,y)$ in harmonic G_γ, we can prove that

$$d = \lim_{n\to\infty}\left[2\int_B \alpha_n(x,0)dx + v^2\iint_G ydxdy + \omega\iint_{G_n}(\psi_n - 1)dxdy\right] = J(\psi,\gamma),$$

where $d = \inf \ J(\psi,\gamma)$ on R. At last, using Schiffer's method of interior variations [3], we can show that $\partial u/\partial n = v$ almost everywhere on the part of γ, lying inside G.

Lemma 4. *Let us suppose that there is (7) and we also have inequalities:*

$$v < \frac{1}{c^2}\left(2 + \frac{\omega c^3}{3}\right),\ \omega\ mes\ G + \frac{2a}{c^2}\left(1 - \frac{\omega c^3}{3}\right) < v\int_0^a \sqrt{1 + g_x^2 dx}. \tag{8}$$

Then G_γ cannot coincide with G, that is $G_\gamma \subset G$, and all the points $(x,y) \in \gamma$ are above the horizontal y=c, except point (0,c).

Proof. Let us suppose that $G_\gamma = G$. Then we have

$$\int_S \frac{1}{y}\frac{\partial\psi}{\partial n}ds = 2\int_B \alpha(x,0)dx + \omega\iint_G dxdy.$$

Now, using $\psi(x,y) = y^2\alpha(x,y) \le y^2(1-\omega c^3/3)/c^2 + \omega y^3/3,\ \alpha(x,y) \le (1-\omega c^3/3)/c^2 + \omega y/3$ for $(x,y) \in \Pi_0$, we obtain inequality

$$v\int_0^a \sqrt{1+g_x^2}dx \le \omega\ mes\ G + \frac{2a}{c^2}\left(1 - \frac{\omega c^3}{3}\right).$$

There is a contradiction, so $G_\gamma \subset G$.

Let $\gamma_0 = (0 \le x \le a, y = c)$ and $\gamma \cap \gamma_0 = \{0 \le x \le a, y = c\}$(here $G_{\gamma_0} \subset G_\gamma$). Then $\psi_0(x.y) = y^2(1-\omega c^3/3)/c^2 + \omega y^3/3$ is the solution of (1)-(4) in $G_{\gamma 0} = (0 < x < a, 0 < y < c)$. According to the maximum principle $\psi = \psi_0$ in $\overline{G}_{\gamma 0}$. Thus, $J(\psi_0,\gamma_0) = J(\psi,\gamma) = d$. Now we compute the first variation J on pair (ψ_0,γ_0), provided that $\overrightarrow{\delta z}=0$ in points (*0*,*c*) and (*a* ,*c*), $(\overrightarrow{\delta z},\vec{n})\geq 0$ and the value $\max\left|\overrightarrow{\delta z}\right|$ is small (see (6)):

$$\delta J\left(\psi,G_\gamma,\delta\psi,\overrightarrow{\delta z}\right)=\int\limits_{\gamma 0}\left\{v^2c^2-\left[\frac{2}{c}\left(1-\frac{\omega c^3}{3}\right)+\omega c^2\right]^2\right\}\frac{\left(\overrightarrow{\delta z},\vec{n}\right)}{c}ds<0\,.$$

Then we obtain the domain $\tilde{G}_\gamma$ with the free boundary $\tilde{\gamma}$, so that $J(\bar{\psi},\bar{\gamma})<J(\psi,\gamma)=d$, where $\tilde{\psi}$ is the solution of (1)-(4) in $\tilde{G}_\gamma$. Let G_γ^* be the symmetrization of $\tilde{G}_\gamma$ domain in relation to axis *y* and let ψ^* be the solution of (1)-(4) in G_γ^*. Then, according to Lemma 3, $J(\psi^*,\gamma^*)\leq J(\tilde{\psi},\tilde{\gamma})$ (here γ^* is the free boundary of G_γ^*). We choose curve γ' so that $G_\gamma^*\subset G_{\gamma'}$. Since $\psi^*(x,y)-\psi'(x,y)\geq 0$ in $\tilde{G}_{\gamma*}$, using Lemma 3,we obtain inequality

$$d\leq J\left(\psi',\gamma'\right)\leq J(\psi^*,\gamma^*)+v^2b\,mes\left(G_{\gamma'}\setminus G_{\gamma^*}\right)<d$$

for small value $mes\left(G_{\gamma'}\setminus G_{\gamma^*}\right)$. Thus, we have a contradiction, since $\left(\psi',\gamma'\right)\in R$. The other cases are similar. The lemma is proved.

Let $z_0=x_0+iy_0$ be any interior point of γ and $K_r:\left|z-z_0\right|<r$, so that $K_r\subset G$ is for small r. Using methods [3], [7], we will prove the existence of the analytic function $g(t)$, $t=\xi+i\eta$ in domain $G_\gamma\cap K_r$, witch is continuous in $\overline{G_\gamma\cap K_r}$ and $g(t)=\bar{t}$ on γ. Let $\omega=\omega(t)$ be the conformal mapping of $G_\gamma\cap K_r$ on the upper half-plane. According to Schwartz's principle, the functions $\Phi_1(t)=g(t)+\bar{t}$ and $\Phi_2(t)=g(t)-\bar{t}$ can be analytically continued through the segments of $w-$plane, which conform γ. Then

$$t(w)=\frac{\Phi_1-\Phi_2}{2}$$

is analytically continued there too. Thus, γ is an analytic arc.

Now we can prove the theorem.

Theorem 1. *Let S:* $y=g(x), a\leq x\leq b,$ *where* $g(x)\in C^2[0,a]$ *and g(x) is the monotonically increasing function, such that* $g(0)=c, g(a)=b, c<b, g'(0)=0, g'(a)=0.$ *Suppose that we have (6), (7) too. Then there exists* $\left(\psi,\gamma\right)-$ *the unique solution of (1)-(5) –, where* γ *is the monotonically increasing arc, analytical in the neighborhood of each of its points, lying inside G,* $\psi(x,y)$ *is the continuous functional in* $\overline{G}_\gamma$, $\psi(x,y)$ *is the continuously differentiable function in* $\overline{G_\gamma}$ *except, perhaps, (a,h),* $c<h\leq b$ *((a,h) which is the end of* γ *).*

Proof. Let $\left(\psi,\gamma\right)$ be the limit pair. We consider only the case when $\gamma\cap\Gamma_2=\left\{x=a, h\leq y\leq b\right\}, c<h<b.$ Let γ_0 be the part of γ, such that points (0,*c*) and (*a*, *h*) are the ends for γ_0. Evidently that $J(\psi,\gamma)=J(\psi,\gamma_0)=d.$ Then we consider the

minimum problem of the functional J on R_h and show that $d_h = J(\psi,\gamma_0)$, where $d_h = \inf J$ on R_h. We have $(\psi,\gamma_0)\in R_h$ and $d_h \le J(\psi,\gamma_0) = d$. Now suppose that (ψ_1,γ_1) is any admissible pair, i.e. $(\psi_1,\gamma_1)\in R_h$. Let us choose $(\psi_2,\gamma_2)\in R$, so that $G_{\gamma_1}\subset G_{\gamma_2}$. We can show that

$$d \le J(\psi_2,\gamma_2) \le J(\psi_1,\gamma_1) + v^2 b\, mes(G_{\gamma_2}\setminus G_{\gamma_1})$$

(see Lemma 4). Then $d \le d_h + \varepsilon$, where $\varepsilon = v^2 b\, mes(G_{\gamma_2}\setminus G_{\gamma_1})$. Letting $\varepsilon \to 0$, we conclude that $d \le d_h$. Thus, $d_h = J(\psi,\gamma_0)$. Evidently (ψ,γ_0) is the solution of (1)-(5). The uniqueness of solution (ψ,γ) follows from [4] (here we also use (9)). Thus, the theorem is proved.

5. Approximate solution. In this section we suppose, that solution (ψ,γ) of (1) – (5) satisfies the conditions: $\psi(x,y)\in C^1(\overline{G_\gamma}),\gamma$ and S have the finite number of common points. Using the Friedrichs transformation [4], we obtain

$$J_1(z) = \iint_\Delta \left[(z_x + \frac{g_x}{g} z)^2 + \frac{1}{g^2} + v^2 g^2 z^2 z_\varphi^2 + 2wg(\varphi-1) z z_\varphi^2 \right] \frac{dxd\varphi}{zz_\varphi},$$

where $\Delta = (0<x<a, 0<\varphi<1), \varphi(x,z) = \psi(x, zg(x)), z(x,\varphi)$ is the solution of equation $\varphi(x,z) - \varphi = 0$. We will study a minimum problem of the functional $J_1(z)$ with a constant domain of integration on the following set:

$$D_z = \{ z : z \in C(\overline{\Delta}), \sqrt{\varphi} z_\varphi \in C(\overline{\Delta}), z(0,1)=0, z(x,0)=0, \min_{(x,\varphi)\in\overline{\Delta}} \sqrt{\varphi} z_\varphi > 0 \}.$$

We define $z_0(x,\varphi)$ as the solution of equation $z(x,\varphi) - z = 0$, where $\varphi(x,z) = \psi(x, zg(x))$ and (ψ,γ) is the solution of(1)-(5). Evidently, that $z_0 \in D_z$ and $z_0(x,\varphi) = \sqrt{\varphi}\eta(x,\varphi), (x,\varphi)\in\overline{\Delta}$, where $\eta(x,\varphi)$ is the smooth function, such that $\eta(x,0)\ne 0, 0\le x\le a.$

Lemma 5. The function $z_0(x,\varphi), (x,\varphi)\in\overline{\Delta}$ is a minimum for $J_1(z)$ *on* D_z.

Proof. Let us define $w(x,\varphi)$ according to the formula $\omega = \ln z$. Then we compute,

$$J_2(w) = \iint_\Delta \left[(w_x + \frac{g_x}{g} z)^2 + \frac{e^{-2w}}{g^2} + v^2 g^2 e^{2w} w_\varphi^2 + 2wg(\varphi-1) e^w e_\varphi^2 \right] \frac{dxd\varphi}{w_\varphi},$$

$$\begin{aligned} \int_0^1 (1-\varepsilon) \frac{d^2 J_2(w_\varepsilon)}{d\varepsilon^2} d\varepsilon &= 2\int_0^1 (1-\varepsilon) \iint_\Delta \{ \left[w_{\varepsilon\varphi}\delta w_x - \delta w_\varphi (w_{\varepsilon x} + \frac{g_x}{g} \right]^2 + \\ &+ \frac{e^{-2w_\varepsilon}}{g^2} [w_{\varepsilon\varphi}^2 \delta w^2 + (w_{\varepsilon\varphi}\delta w + \delta w_\varphi)^2] \} \frac{d\varepsilon dx d\varphi}{w_{\varepsilon\varphi}^3} + \\ &+ 2v^2 \int_0^1 (1-\varepsilon) \int_0^a g^2(x) e^{2w_\varepsilon(x,1)} \delta w^2(x,1) d\varepsilon dx, \end{aligned} \qquad (9)$$

where $J_1(z) = J_1(e^w) = J_2(w),\ w_\varepsilon = w_0 + \varepsilon\delta w,\ \delta w = w - w_0,\ w_0 = \ln z_0,\ \delta z = z - z_0 = z_0 \delta w,$

$0 \le \varepsilon \le 1$, z is any element of D_z. Now, using the Friedrich formula [4]

$$J_2(w) = J_2(w_0) + \frac{d}{d\varepsilon} J_2(w_\varepsilon)|_{\varepsilon=0} + \int_0^1 (1-\varepsilon) \frac{d^2 J_2(w_\varepsilon)}{d\varepsilon^2} d\varepsilon, \qquad (10)$$

we conclude that $J_1(z_0) = J_2(w_0) \le J_2(w) = J_1(z)$ for every $z \in D_z$. Thus the lemma is proved.

We will minimize $J_1(z)$ on D_z, using the polynomials

$$z_n(x,\varphi,a_{kj}) = z_n(x,\varphi) = \sqrt{\varphi} \sum_{k=0}^{m} \sum_{j=0}^{m_k} a_{kj} x^j \varphi^k, \;\; n = \sup_{0 \le k \le m} \left(k + m_k\right),$$

where $a_{kj} \in E_r$ (the Euclidian space),

$$r = \sum_{k=0}^{m} (m_k + 1), \qquad D_r = E_0^r \cap D_r^+, \qquad E_0^r : \sum_{k=0}^{m} a_{k0} - 1 = 0,$$

$$D_r^+ = \left\{ a_{kj} : \min_{(x,\varphi) \in \bar{\Delta}} \sqrt{\phi} z_{n\varphi} > 0 \right\}.$$

We will seek a_{kj} as a solution of nonlinear Ritz system equations (see [8], [9], too).

$$\frac{\partial J_3(a_{kj})}{\partial a_{p0}} + \lambda = 0 \quad \text{p=0,1,2,…m;}$$

$$\frac{\partial J_3(a_{kj})}{\partial a_{p0}} = 0, \text{ q=1,2,… } m_p\text{; p=0,1,…m;} \qquad (11)$$

$$\sum_{k=0}^{m} a_{k0} - 1 = 0, \qquad J_3(a_{kj}) = J_1(\sqrt{\varphi} \sum_{k=0}^{m} \sum_{j=0}^{m_k} a_{kj} x^j \varphi^k).$$

Theorem 2. *There exists a minimum* $a_{kj}^* \in D_r$ of the functional J_3 on D_r, *where* a_{kj}^* *is an interior finite point in* D_r.

Proof. Let it be sequence $a_{kjp} \to \infty$ when $p \to \infty$, *i.e.* a_{kjp} are unbounded in E_r. Let us define $J_4(c_{kjp}) = J_3(M_p c_{kjp})$, where

$$M_p^2 = \sum_{k=0}^{m} \sum_{j=0}^{m_k} a_{kjp}^2, \;\; c_{kjp} = \frac{a_{kjp}}{M_p}.$$

Then we can show, that $J_3 \to \infty$, if $p \to \infty$. On the other hand $J_3(a_{kj})$ is unbounded function on ∂D_r. So the theorem is proved.

Remark. Now, using the Lagrangian function, we conclude that a_{kj}^* is a solution of systems (11). Thus, there exists a solution of (11) for every fixed n.

6. Regularity. At first, we define $z_n(x,\varphi;a_{kj}^*) = z_n^*$.

Lemma 6. *The sequence* z_n^* *is the minimizing sequence for* J_1 *on* D_z.

Proof. Let us define ||z|| according to the formula

$$\| z \| = \max | z | + \max | z_x | + \max | \sqrt{\varphi} z_\varphi |,$$

where $(x,\varphi) \in \overline{\Delta}$. For sufficiently small value $\varepsilon > 0$ we can choose the polynomial $P_n\ (x,\varphi)$ so that $\| z_0 - \sqrt{\varphi} P_n \| < \varepsilon,$ where

$$z_n = \sqrt{\varphi} P_n(x,\varphi) \in D_z, \quad P_n = \sum_{k=0}^{m} \sum_{j=0}^{m_k} b_{kj} x^j \varphi^k.$$

Then we obtain $J_1\,(z_n^*) - d \le J_1(z_n) - d = J_1(z_n) - J_1(z_0) < \tilde{\varepsilon}$, here $d = J_1(z_0) \le J_1(z\ )$ for every $z \in D_z$ (see Lemma 5) and $\tilde{\varepsilon} > 0$ is sufficiently small. Thus the lemma is proved.

The function $y_n(x) = g(x) z_n(x,1,a_{kj}^*), \quad 0 \le x \le a$ is an approximate equation of free boundary γ in problem (1)-(5).

Theorem 3. *The sequence* $z_n(x,1,a_{kj}^*) \to z_0(x,1)$ *strongly in* $L_2(0,a)$.

Proof. According to Lemma 6 $\varepsilon_n = J_1(z_n^*) - J_1(z_0) \to 0$, when $n \to \infty$.

Taking into consideration (10), we obtain the inequality

$$\int_0^1 (1-\varepsilon) \frac{d^2 J_2(w_\varepsilon)}{d\varepsilon^2} d\varepsilon \le \varepsilon_n ,$$

which can be written, as

$$2v^2 \int_0^1 (1-\varepsilon) \int_0^a g^2(x) e^{2w_\varepsilon(x,1)} \delta w^2(x,1) d\varepsilon dx \le \varepsilon_n ,$$

where $w_\varepsilon = w_0 + \varepsilon(w_n - w_0),$ $w_0 = \ln z_0$, $w_n = \ln z_n^*,$ $0 \le \varepsilon \le 1.$ Then we get the estimation

$$\int_0^a \delta w^2(x,1) dx \le \frac{2\varepsilon_n}{v^2 c^2}, \qquad \delta z = z_n^* - z_0, \qquad (\delta z = \delta w\ exp\ w_0).$$

Thus $\| z_n(x,1,a_{kj}^*) - z_0(x,1) \|_{L_2(0,a)} \to 0, \quad when \quad n \to \infty$.

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RESUME

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Axially Symmetric Flow

Background: A new class of problems with a free boundary and hydrodynamic origin is investigated, when on part of the free boundary the nonlinear Bernoulli's law is set in kind of inequality. The existence of a classical solution of respective nonlinear problem is proved. The free boundary proves to be an analytic monotonic curve. The proof is underlain by the Schiffer's method of interior variations and Steiner symmetrization.

Materials and methods: This paper is aimed at development of exact and approximate methods of solution of nonlinear problems with a free boundary. The following methods are devised:

– justification of correctness of class of free boundary problems with variational nature;

– construction of approximate solution of such nonlinear problems by the Ritz method, and investigation of their convergence.

Results: The convergence of the Ritz approximation to the exact solution in the integral metric is ascertained. The developed methods can be applied for studying of the whole class of free boundary problems with variational nature.

Conclusion: The paper proves the existence of a classical solution of the new class of free boundary problems of the Bernoulli's type. The approximate solution converging metrically to the exact one in L_2 is constructed.

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